(1) Define following with example –

Dr. Sunil Jee
S.B. College

Def^n (Homogeneous Equation):– An equation of the type $M + N\frac{dy}{dx} = 0$, where M, N are homogeneous function of x & y and of the same degree is called a homogeneous differential equation. Such equation can be solved by the substitution $y = vx$ and therefore,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$ in the given equ^n.

Example:– $$\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$$

Solu^n– let us Put $\frac{y}{x} = v$, i.e. $y = vx$, So that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\therefore \cancel{v} + x\frac{dv}{dx} = \cancel{v} + \tan v$$

~~or, $\frac{dx}{x} = \frac{dv}{\tan v} = \frac{\cos v\, dv}{\sin v} = d(\log \sin v)$.~~

~~Integrating, we have~~

~~$\log x + \log$~~

$$\text{or, } x\frac{dv}{dx} = \tan v$$

$$\text{or, } \frac{dv}{\tan v} = \frac{dx}{x}$$

$$\text{or, } \cot v\, dv = \frac{dx}{x}$$

on, Integrating, we have

$$\int \cot v\, dv = \int \frac{dx}{x}$$

$$\text{or, } \log \sin v = \log x + \log c$$

$$\text{or, } \log \sin\frac{y}{x} = \log xc$$

$$\therefore \sin\frac{y}{x} = xc$$

~~required Solution~~
required Solution.

(Defn) <u>orthogonal trajectory</u> :- A curve which cuts every number of a given family of curves according to a given law is called a trajectory of the given family of curves. when each number of trajectory cuts every number of the given curves at right angles, then the trajectory is called <u>orthogonal</u>.

<u>Example</u>:- Find the orthogonal trajectory of $y = ax^2$

Soln:- $\because y = ax^2$ —— ①

Differentiating with respect to x, we have

$$\frac{dy}{dx} = 2ax,$$

$$\text{or, } a = \frac{1}{2x} \cdot \frac{dy}{dx}$$

Substituting the value of a in eqn ①, we have

$$y = \frac{1}{2x} \cdot \frac{dy}{dx} \cdot x^2$$

For orthogonal trajectory substituting

$$-\frac{dx}{dy} \text{ for } \frac{dy}{dx}$$

$$\therefore y = \frac{1}{2x} \cdot \left(-\frac{dx}{dy}\right) x^2$$

$$\text{or, } 2y\,dy = -x\,dx$$

on integrating, we have

$$y^2 = -\frac{x^2}{2} + \frac{c^2}{2}$$

$$\text{or, } 2y^2 = -x^2 + c^2$$

$$\text{or, } x^2 + 2y^2 = c^2.$$

Defn. Singular Solution:- The equation $y = px + f(p)$ — ①

Where $p = \frac{dy}{dx}$ and $f(p)$ is any function of p, is known as Clairaut's Equation.

To solve this equn., diff. with r. to x, we have

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

or, $\left[x + f'(p)\right]\frac{dp}{dx} = 0$

Therefore, $\frac{dp}{dx} = 0$ or, $x + f'(p) = 0$

when $\frac{dp}{dx} = 0$, $dp = 0$

on integration $p = C$, where C = an arbitrary Constt.

Substituting in ①, the required solution is

$$y = Cx + f(C)$$

If we eliminate p between differential equation $x + f'(p) = 0$ and the given $y = px + f'(p)$ we get solution which does not contain any arbitrary constant, such a solution is called a Singular Solution.

Example:- Solve $y = px + \frac{a}{p}$ and obtain the singular solution.

Soln. The equn. is of the form $y = px + f(p)$.

So it is a Clairaut's eqn.

∴ the general solution is obtained by putting $p = C$, in it. Hence the general solution is

$y = Cx + \frac{a}{C}$ where C is the arbitrary Constt.

For singular solution of $y = xp + \frac{a}{p}$ — ①

Diff. ① w.r.t. p, we have

$$0 = x - \frac{a}{p^2} \quad \text{— ②}$$

Eliminating p between ① & ②

$$y^2 = \left(xp + \frac{a}{p}\right)^2 = x^2p^2 + \frac{a^2}{p^2} + 2ax$$

$$= x^2 \cdot \frac{a}{x} + a \cdot x + 2ax \quad \left[\because p^2 = \frac{a}{x} \text{ from } (2)\right]$$

$$\text{or, } y^2 = 4ax$$

$\therefore$ The Singular Solution is $y^2 = 4ax$.

Defn. (Complementary function):- Let us Consider a general linear equation of the Second order,

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = V \quad \text{--- } (1)$$

where P & Q are Constt. and V is a given function of x.

The Complete Soln. of (1) may be written as,

$$y = u + w \quad \text{--- } (2)$$

where w is any function whatever which Satisfies (1) as it Stands and u is the general Solution of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad \text{--- } (3)$$

which differs from (1) by the absence of the right-hand member.

Then u & w are Called the "Complementary function" and the Particular integral respectively. The Complementary function must be the most general Solution of (3) and would involve two arbitrary Constt.

Example:- $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = x^2$.

Soln. We have $(D^2 + D + 1)y = x^2$

$\therefore$ The Auxiliary equn. $m^2 + m + 1 = 0$

$\therefore m = \frac{-1 \pm \sqrt{3}i}{2}$

$\therefore$ The Complementary function

$$= e^{-\frac{1}{2}x}\left(A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x\right)$$

Now, the P.I. $= \frac{1}{D^2+D+1} x^2$

$$= \frac{D-1}{D^3-1} x^2 = -(D-1)(1-D^3)^{-1} x^2$$

$$= -[D-1)(1+D^3+D^6+D^9+\cdots)] x^2$$

$$= -[D-1+D^4-D^3+\cdots] x^2$$

$$= x^2 - 2x$$

$\therefore$ The Complete Solution is

$$y = e^{-x/2}\left(A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x\right) + x^2 - 2x.$$